$\Sigma b_i{}^r = k'b_1{}^r + \delta' k' b_1{}^r$, where δ and δ' can be made as small in absolute value as we please by taking r sufficiently large. Since the left-hand sides of these equations are equal, the right-hand sides must be equal also. Equating them we obtain

$$\frac{k}{k'} = \frac{b_1{}^r}{a_1{}^r}\frac{(1+\delta')}{(1+\delta)} = \frac{b_1{}^r}{a_1{}^r}(1+n),$$

where n is an infinitesimal. If, now, a_1 and b_1 are not equal we can make the right-hand side as small (or as large) as we please by taking r sufficiently large. This is impossible for the ratio $k \div k'$ is a definite number. Consequently $a_1 = b_1$, and $k = k'$. In each equation we may now subtract from each side the a's and b's proven equal and proceed as before to prove the largest remaining a's and b's to be equal in value and number. We may continue in this way until there are no a's (or b's) left, when the b's (a's) must also be exhausted.

471 was also solved by C. F. Gummer and Frank Irwin.

473. Proposed by J. I. GINSBURG, Student, Cooper Union, New York.

Factor the expression $x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1$.

Solution by Claribel Kendall, University of Colorado.

Multiplying the given expression by $x^5 - 1$, we obtain $x^{35} - 1$. This may be factored in the following way:

First, consider

$$\begin{aligned}(1)\qquad x^{35} - 1 &= (x^5)^7 - 1 = (x^5 - 1)(x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1)\\ &= (x-1)(x^4 + x^3 + x^2 + x + 1)(x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1).\end{aligned}$$

Next consider

$$\begin{aligned}(2)\qquad x^{35} - 1 &= (x^7)^5 - 1 = (x^7 - 1)(x^{28} + x^{21} + x^{14} + x^7 + 1)\\ &= (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)(x^{28} + x^{21} + x^{14} + x^7 + 1).\end{aligned}$$

The factors of $x^{35} - 1$ must be the same in both (1) and (2). $x - 1$ occurs in both. The prime factor $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ in (2) must also be found in (1). It cannot be contained in the prime factor $x^4 + x^3 + x^2 + x + 1$ and hence must be found in the third factor. By division the factors of

$$x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1$$

are found to be

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

and

$$x^{24} - x^{23} + x^{19} - x^{18} + x^{17} - x^{16} + x^{14} - x^{13} + x^{12} - x^{11} + x^{10} - x^8 + x^7 - x^6 + x^5 - x + 1.$$

Also variously solved by H. C. Feemster, Horace Olson, O. S. Adams, W. F. Shelton, J. W. Baldwin, A. W. Smith, E. B. Escott, L. C. Mathewson, Paul Capron, E. J. Oglesby, and the Proposer.

GEOMETRY.

493. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.

Construct three circles each of which shall be tangent to the other two and to two sides of a given triangle.

Note by J. L. Coolidge, Harvard University.

It is always a pleasure to see this old friend. Few problems in elementary geometry have a longer or more distinguished pedigree. At least forty articles dealing with it were published in the nineteenth century and the twentieth is doing its share.[1] The history is briefly this. The

[1] Simon, *Die Entwickelung der Elementargeometrie im XIX Jahrhundert*, Leipzig, 1906, pp. 147ff.

problem was first proposed and solved in 1803 by Malfatti, and is generally known by his name. A very beautiful construction was given by Steiner in 1826 with no proof, but the appended remark that this construction showed how powerful were the methods which the writer had developed.[1] Steiner's construction served as a challenge to geometers, and many proofs were attempted. By far the simplest and most elegant was given in 1856 by Hart.[2] The German mathematicians were ill pleased that the first good proof should be in English; perhaps they failed to appreciate the strategic significance of the fact that Hart was an Irishman. At any rate they objected that Hart had used methods unknown to Steiner.[3] This criticism seems to us trivial. Steiner was a sly old fox, who probably knew a good deal more mathematics than he ever put on paper. For instance, there seems good reason to believe that he was familiar with inversion in a circle, though he did not give it as one of his working methods.[4] As for simplicity, one has but to compare Hart's proof as given, let us say, in Casey's *Sequel to Euclid*,[5] with the proof in Petersen's little classic on geometrical constructions[6] to see how immeasurably superior the former is. Analytic determinations of the radii and points of contact of the circles have not been wanting; that given by Professor Gummer in the MONTHLY, Vol. XXIV, No. 3, being quite as simple as any. On the other hand the Lemoine geometrographic numbers called for by Steiner's construction can be reduced to the remarkably small proportions of Simplicity 66, Exactitude 42.[7]

502. Proposed by R. P. BAKER, University of Iowa.

A designer of machinery requires a curve having the following properties:

(1) A closed curve touching a given circle at two diametral points and enclosing it.

(2) The sum of the three radii from the center of this circle to the curve which make with each other angles of 120° is constant.

(3) The locus of a point which lies at some constant distance from the curve on its inner normal must be such that it is also the locus of a point fixed on a bar of some simple linkage. In estimating the value of the word "simple" pivoted bars are preferred to slides and the total number should be as small as possible.

Condition (3) is needed to enable a cylinder to be ground accurately to the curve.

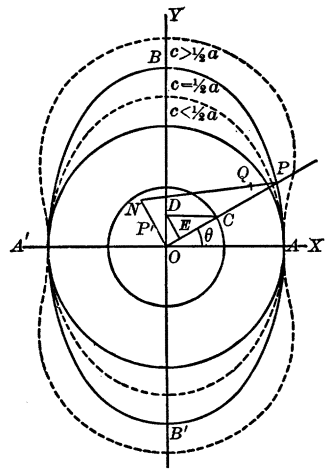

FIG. 1.

SOLUTION BY TOBIAS DANTZIG, Indiana University.

Let (Fig. 1) AOA' be the diameter of contact and BOB' a perpendicular diameter. I shall seek a continuous algebraic curve symmetric with respect to the two diameters satisfying the conditions of the problem. If $\rho = f(\theta)$ is the equation of the curve, f is a function with a period π in θ and consequently of the form

$$\rho = a + c\sin^2\theta + e\sin^4\theta + \cdots, \tag{1}$$

where a is evidently the radius of the circle. If we stop at the first term, we obtain the trivial solution of the circle itself, which evidently satisfies all the conditions of the problem. If we take two terms,

$$\rho = a + c\sin^2\theta, \tag{2}$$

we obtain a circular sextic whose cartesian equation is

$$(x^2+y^2)^3 = (ax^2+by^2)^2, \tag{3}$$

[1] "Einige geometrische Betrachtungen," *Crelle*, Vol. 1, 1826.

[2] "Geometrical Investigation of Steiner's Solution of Malfatti's Problem," *Quarterly Journal of Mathematics*, Vol. 1, 1856.

[3] SCHROETER, *Crelle*, Vol. LXXVII, 1874, p. 232.

[4] BÜTZBERGER, *Ueber bizentrische Polygone*, Leipzig, 1913, pp. 50ff.

[5] P. 149 in the first ed., Dublin, 1881.

[6] Fourth edition of the French translation, *Méthodes et théories pour la résolution des problèmes de constructions geometriques*, Paris, 1908, pp. 103ff.

[7] HAGGE, "Zur Konstruktion der Malfattiscen Kreise," *Zeitschrift für mathematische Unterricht*, Vol. XXXIX, 1908, p. 588.